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Paths and Edge-Connecivity in Graphs

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1. INTRODUCTION

We consider finite undirected graphs passibly with multiple edges but without loops. Let G be a graph and let V(G) and E(G) be the sets of vertices and edges of G respectively. For two distinct vertices x and y, let $\lambda_G(x,y)$ be the maximal number of edge-disjoint paths between x and y, and let $\lambda_G(x,x)=\infty$. For an integer $k\geq 1$, let $\Gamma(G,k)$ be

$$\{X\subseteq V(G) \mid \text{For each } x,y\in X,\ \lambda_G(x,y)\geq k\}.$$

Let $(s_1,t_1),\dots,(s_k,t_k)$ be pairs of vertices of G. When is the following statement true ?

(1.1) There exist edge-disjoint paths $P_1,\dots,P_k$ such that P_i has ends s_i, t_i $(1\leq i\leq k)$.

Seymour [8] and Thomassen [9] characterised such graphs when k=2, and Seymour [8] when $|\{s_1,\dots,s_k,t_1,\dots,t_k\}|=3$.

For integers $k\geq 1$ and $n\geq 2$, set

$$g(k)=\min\{m \mid \text{If } G \text{ is } m\text{-edge-connected, then (1.1) holds}\},$$

$$\lambda'(k,n)=\min\left\{ m \;\middle|\; \begin{array}{l} \text{If } |\{s_1,\dots,s_k,t_1,\dots,t_k\}|\leq n \text{ and} \\ \{s_1,\dots,s_k,t_1,\dots,t_k\}\in\Gamma(G,m), \text{ then (1.1)} \\ \text{holds} \end{array} \right\},$$

$$\lambda(k,n)=\min\left\{ m \;\middle|\; \begin{array}{l} \text{If } |\{s_1,\ldots,s_k,t_1,\ldots,t_k\}| \le n \text{ and} \\ \lambda_G(s_i,t_i) \ge m \ (1 \le i \le k), \text{ then (1.1) holds} \end{array} \right\},$$

and set

$$\lambda'(k)=\lambda'(k,2k)=\lambda'(k,m) \ (m > 2k) \text{ and } \lambda(k)=\lambda(k,2k).$$

Then for each $k \ge 1$,

$$\lambda'(k,3)=\lambda(k,3) \text{ and } \lambda(k) \ge \lambda'(k) \ge g(k) \ge k.$$

For $n \ge 4$ and even integer $k \ge 2$,

$$g(k) > k \quad \text{and} \quad \lambda(k) \ge \lambda(k,n) \ge \lambda'(k,n) > k$$

(see Figure 1 in which $k/2$ represents the number of parallel edges).

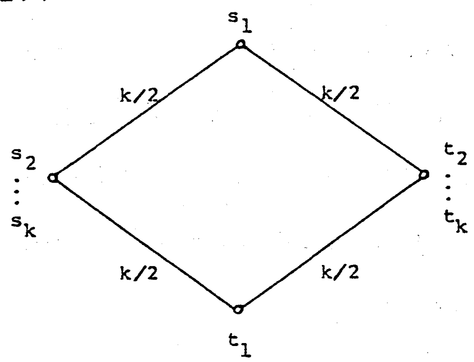

Figure 1.

Thomassen [9] gave following Conjecture 1, and we give following Conjecture 2 slightly stronger than Conjecture 1.

CONJECTURE 1. For each integer $k \ge 1$,

$$g(k)=\begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{cases}.$$

CONJECTURE 2. For each integer $k \ge 1$,

$$\lambda(k)=\begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{cases}.$$

It easily follows from Menger's theorem that $\lambda(k)\leq 2k-1$; thus $\lambda(1)=1$ and $\lambda(2)=3$. Cypher [1] proved $\lambda(4)\leq 6$ and $\lambda(5)\leq 7$, and $\lambda(3)=3$ was announced in [5] and proved in [6] by the author. Enomoto and Saito [2] proved g(4)=5, and independently Hirata, Kubota and Saito [3] proved $\lambda(k)\leq 2k-3$ $(k\geq 4)$.

Our main results are the following.

THEOREM 1. Suppose that $k\geq 2$ is an integer, G is a graph, $\{a_1,a_2\}\subseteq T\subseteq V(G)$, $|T|\leq 3$ and $T\in\Gamma(G,k)$. Then there exists a path P between a_1 and a_2 such that $T\in\Gamma(G-E(P),k-1)$.

THEOREM 2. Suppose that $k\geq 5$ is an odd integer, G is a graph, $\{a_1,a_2,a_3\}\subseteq T\subseteq V(G)$, $a_i\neq a_j$ $(1\leq i<j\leq 3)$, $|T|\leq 5$ and $T\in\Gamma(G,k)$. Then

(1) If $|T|\leq 4$, then there exists a path P between a_1 and a_2 such that $T\in\Gamma(G-E(P),k-1)$.

(2) For m=2,3 if $|T|\leq 4$ and for m=3 if $|T|=5$, there exist edge-disjoint paths P_1 between a_1 and a_2 and P_2 between a_1 and a_m such that $T\in\Gamma(G-\bigcup_{i=1}^{2}E(P_i),k-2)$.

THEOREM 3. For each integer $k\geq 1$,

$$\lambda(k,3)=k \text{ and } \lambda(k,4)=\lambda(k,5)=\begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even.} \end{cases}$$

In Theorem 2(2) if m=2 and $|T|=5$, then the concluson does not always hold. Figure 2 gives a countrexample with with k=7.

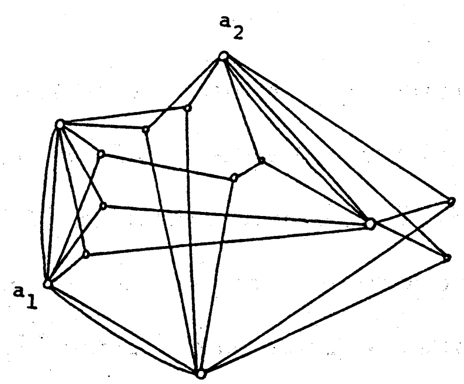

Figure 2.

When k is odd and $|\{s_1,\dots,s_k,t_1,\dots,t_k\}| \geq 4$, if for some $1 \leq i \leq k$,

$$\lambda_G(s_i,t_i) < k,$$

then (1.1) does not always hold. Figure 3 gives a counterexample.

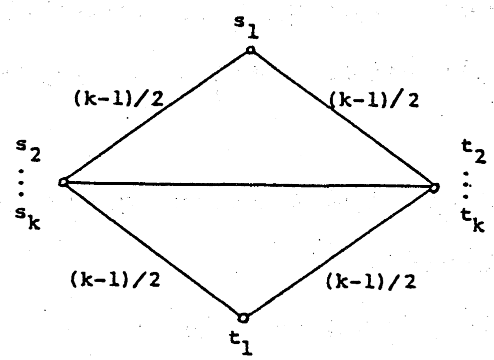

Figure 3.

Notations and Difinitions. Let $X,Y \subseteq V(G)$, $F \subseteq E(G)$, $\{x,y\} \subseteq V(G)$ and $e \in E(G)$. We often denote $\{x\}$ by x and $\{e\}$ by e. The subgraph of G induced by X is denoted by $\langle X \rangle_G$ and the subgraph obtained from G by deleting X (F) is denoted by G-X (G-F). $\partial_G(X,Y)$ denotes the set of edges with one end in X and the other in Y, and $\partial_G(X)$ denotes $\partial_G(X,V(G)-X)$. $\lambda_G(X,Y)$ denotes the maximal number of edge-disjoint paths with one end in X and the other in Y. $\partial_G(X)$

is called an n-cut if $|\partial_G(X)|=n$ and $\langle X\rangle_G$ and $\langle V(G)-X\rangle_G$ are both connected. An n-cut $\partial_G(X)$ is called nontrivial if $|X|\geq 2$ and $|V(G)-X|\geq 2$, trivial otherwise. $d_G(x)$ denotes the degree of x and $N_G(x)$ denotes the set of vertices adjacent to x. We regard a path and a cycle as subgraphs of G. A path $P=P[x,y]$ denotes a path between x and y, and for $x',y'\in V(P)$, $P(x',y')$ denotes a subpath of P between x' and y'.

2. PROOF OF THEOREM 1

For a vertex $w\in V(G)$ and $b,c\in N_G(w)$, we let $G_w^{b,c}$ be the graph $(V(G),E(G)\cup e-\{f,g\})$, where e is a new edge with ends b and c, $f\in\partial_G(w,b)$ and $g\in\partial_G(w,c)$. We require the following lemmas.

LEMMA 2.1 (Mader [4]). Suppose that w is a non-separating vertex of a graph G with $d_G(w)\geq 4$ and with $|N_G(w)|\geq 2$. Then there exist $b,c\in N_G(w)$ such that for each $x,y\in V(G)-w$,

$$\lambda_{G_w^{b,c}}(x,y)=\lambda_G(x,y).$$

Now we prove Theorem 1 by induction on $|E(G)|$. We may assume that $a_1\neq a_2$ and $|T|=3$. If G has a nontrivial k-cut $\partial_G(X)$ $(X\subseteq V(G))$ separating T, then let H (K) be the graph obtained from G by contructing $V(G)-X$ (X) to a new vertex u (v). Set $T_H=(X\cap T)\cup u$ and $T_K=(T-X)\cup v$. We may let $|T\cap X|=2$. By induction for H and $(T\cap X)\cup u$ instead of for G and T, the result holds. Thus the result follows. Hence we may assume that each edge is incident to a vertex of T.

Case 1. There exists $x\in V(G)-T$.

If $d_G(x)\geq 4$, then by Lemma 2.1 there exists $b,c\in N_G(x)$ such that for each $y,z\in V(G)-x$,

$$\lambda_{G_x^{b,c}}(y,z)=\lambda_G(y,z).$$

By induction the result holds in G_x. Thus wemay assume that $d_G(x)=3$ and clearly that $N_G(x)=T$. Now the path $P[a_1,a_2]$ with $E(P)\subseteq\partial_G(x)$ is a required path.

Case 2. $V(G)=T$.

The result easily follows.

3. PROOF OF THEOREM 2.

We call a graph G is elemental for $V_1\subseteq V(G)$ if $V(G)=V_1\cup V_2$, $V_1\cap V_2=\emptyset$ and for each $x\in V_2$, $d_G(x)=3$, $|N_G(x)|=3$ and $N_G(x)\subseteq V_1$. We call a graph G is elemental for $V_1\subseteq V(G)$ and an integer $k\geq 1$ if G is elemental for V_1 and for each $x\in V_1$, $d_G(x)=k$. For integers $p\geq 0$ and $q\geq 0$, we call a graph G is $G(p,q)$ if G is elemental for some $V_1=\{x_1,x_2,x_3\}\subseteq V(G)$, $|V(G)-V_1|=q$ and $|\partial_G(x_i,x_j)|=p$ $(1\leq i<j\leq 3)$. Let G be an elemenal graph for $V_1\subseteq V(G)$. We call a subgraph S an elemental star if $V(S)\subseteq V_1$, $|V(S)|=2$ and $|E(S)|=1$, or if for some $x\in V(G)-V_1$, $V(S)=N_G(x)\cup x$ and $E(S)=\partial_G(x)$.

We require the following lemmas.

LEMMA 3.1 (Okamura [7]). Suppose that $k \geq 4$ is an integer, G is a graph, $\{s,t\} \subseteq T \subseteq V(G)$ and $T \in \Gamma(G,k)$. Then

(1) For each non-separating edge e incident to s, there exists a path P between s and t passing through e such that

$$T \in \Gamma(G-E(P),k-2) \text{ and } \{s,t\} \in \Gamma(G-E(P),k-1).$$

(2) For each vertex a of $T-\{s,t\}$ with fewer degree than $2k$ and for each edge f incident to a, there exists a path P between s and t not passing through a such that

$$T \in \Gamma(G-E(P),k-2),\ \{s,t,a\} \in \Gamma(G-E(P),k-1),$$

and

$$\{s,a\} \text{ or } \{t,a\} \in \Gamma(G-E(P)-f,k-1).$$

(3) For each non-separating edges e and e' incident to s, there exists a cycle C passing through e and e' such that

$$T \in \Gamma(G-E(C),k-2).$$

LEMMA 3.2 (Okamura [7]). Suppose that $n \geq 4$ is an integer and $k \geq 3$ is an odd integer. If for each odd integer $1 \leq m \leq k$,

$$\lambda'(m,n)=m,$$

then

$$\lambda(k,n)=k \quad \text{and} \quad \lambda(k+1,n)=k+2 .$$

LEMMA 3.3. Suppose that $k \geq 3$ is an integer, G is an elemental graph for $T \subseteq V(G)$ and k, $T \in \Gamma(G,k)$, G has no nontrivial k-cut separating T, and that S_1,S_2,S_3 are elemental stars of G. If $V(S_1) \cap V(S_2) \cap V(S_3)=\emptyset$, then

$T\in\Gamma(G-\bigcup_{i=1}^{3} E(S_i),k-2)$.

Proof. Assume that $X\subseteq V(G)$, $|X|\leq|V(G)-X|$ and X separates T. Set $G'=G-\bigcup_{i=1}^{3}E(S_i)$. If $|X|=1$, then let $X=\{x\}$. Since $d_{G'}(x)\geq d_G(x)-2=k-2$, we have $|\partial_{G'}(X)|\geq k-2$. If $|X|\geq 2$, then $|\partial_G(X)|\geq k+1$, and so $|\partial_{G'}(X)|\geq k-2$. Now Lemma 3.3 is proved.

LEMMA 3.4. Suppose that $k\geq 2$ is an integer, G is an elemental graph for $T=\{x_1,x_2,x_3,x_4\}$ $V(G)$ and k, $|T|=4$ and $T\in\Gamma(G,k)$. Then

(1) One of the following holds.

(i) $\partial_G(x_1,x_2)\neq\emptyset$, $\partial_G(x_1,x_3)\neq\emptyset$, or for some $y\in V(G)-T$, $N_G(y)=\{x_1,x_2,x_3\}$.

(ii) k is even, $|\partial_G(x_2,x_3)|=k/2$, and

$|\{y\in V(G)-T \mid N_G(y)=\{x_i,x_1,x_4\}\}|=k/2$ $(i=2,3)$.

(2) One of the following holds.

(i) For each $1\leq i<j\leq k$, G has an elemental star S containing x_i and x_j.

(ii) k is even and G is the graph obtained from four cycle by replacing each edge by $k/2$ parallel edges.

(3) If G has no nontrivial k-cut separating T, then

(i) $\partial_G(x_1,x_2)\neq\emptyset$ or G has two elemental stars containing x_1 and x_2.

(ii) One of the following holds.

(a) G has edge-disjoint paths $P_1[x_1,x_2]$ and $P_2[x_1,x_3]$ such that for $i=2$ or 4,

$\{x_i, x_3\} \in \Gamma(G-\bigcup_{i=1}^{2} E(P_i), k-1)$ and $T \in \Gamma(G-\bigcup_{i=1}^{2} E(P_i), k-2)$.

(b) For each $e \in \partial_G(x_3)-\partial_G(x_3, x_2)$, G has edge-disjoint paths $P_1[x_1, x_2]$ and $P_2[x_1, x_3]$ such that $e \in E(P_2)$ and $T \in \Gamma(G-\bigcup_{i=1}^{2} E(P_i), k-2)$.

Proof. For $1 \leq i, j \leq 4$, set

$$p_{i,j}=|\partial_G(x_i, x_j)|,$$

$$R_i=\{y \in V(G)-T \mid N_G(y)=T-x_i\},$$

$$r_i=|R_i|.$$

(1) Assume $p_{1,2}=p_{1,3}=r_4=0$. Then

$$d_G(x_1)=k=p_{1,4}+r_2+r_3,$$

$$d_G(x_4)=k=p_{1,4}+p_{2,4}+p_{3,4}+r_1+r_2+r_3$$

Thus

$$p_{2,4}=p_{3,4}=r_1=0.$$

Since $T \in \Gamma(G,k)$, we have

$$|\partial_G(\{x_2, x_3\})|=r_2+r_3 \geq k.$$

Thus

$$p_{1,4}=0.$$

By compairing $d_G(x_i)$ with $d_G(x_j)$ for $1 \leq i < j \leq 3$, we have

$$r_2=r_3=p_{2,3}.$$

Now (ii) follows.

(2) Assume $p_{1,2}=r_3=r_4=0$. Then by compairing $d_G(x_1)+d_G(x_2)$ with $d_G(x_3)+d_G(x_4)$, we have

$$r_1=r_2=p_{3,4}=0.$$

Now by compairing $d_G(x_3)=k=p_{1,3}+p_{2,3}$ with $d_G(x_i)$ for $i=1,2$, we have

$$p_{1,4}=p_{2,3} \text{ and } p_{2,4}=p_{1,3}.$$

Moreover

$$|\partial_G(\{x_1,x_4\})|=p_{1,3}+p_{2,4}=2p_{1,3}\geq k,$$

$$|\partial_G(\{x_1,x_3\})|=p_{1,4}+p_{2,3}=2p_{1,4}\geq k.$$

Thus

$$p_{1,3}=p_{2,3}=p_{2,4}=p_{1,4},$$

and so (ii) follows.

(3) (i) We assume $p_{1,2}=r_4=0$, and then prove $r_3\geq 2$. Since any cut separatong $\{x_1,x_3\}$ and $\{x_2,x_4\}$ or separating $\{x_1,x_4\}$ and $\{x_2,x_3\}$ has more than k edges we have

(3.1) $p_{1,4}+p_{2,3}+p_{3,4}+r_1+r_2+r_3\geq k+1$,

and

(3.2) $p_{1,3}+p_{2,4}+p_{3,4}+r_1+r_2+r_3\geq k+1$.

By compairing $d_G(x_3)+d_G(x_4)$ with (3.1)+(3.2), we have

$$r_3\geq 2.$$

(ii) If there exists $f\in\partial_G(x_1,x_3)$, then by Lemma 2.1 G has a path $P[x_3,x_2]$ such that $f\in E(P)$, $\{x_3,x_2\}\in\Gamma(G-E(P),k-1)$ and $T\in\Gamma(G-E(P),k-2)$, and so (a) follows. Thus we may let

$$p_{1,3}=p_{1,2}=0,$$

then by (1)

$$r_4>0.$$

If $r_3>0$, then for $y_1\in R_4$ and $y_2\in R_3$,
$\{x_3,x_4\}\in\Gamma(G-\bigcup_{i=1}^{2}\partial_G(y_i),k-1)$ and $T\in\Gamma(G-\bigcup_{i=1}^{2}\partial_G(y_i),k-2)$,
and so (a) follows. Thus we may let

$$r_3=0.$$

Then by (1) and (3)

$$p_{1,4} > 0 \text{ and } r_4 \geq 2.$$

Let y be another end of e, then $y=x_4$ or y R_i ($i=1,2$ or 4). In each case (b) easily follows.

LEMMA 3.5. Suppose that $k \geq 3$ is an odd integer, G is a graph, $\{x_1,x_2,x_3\} \subseteq T \subseteq V(G)$, $x_i \neq x_j$ $(1 \leq i < j \leq 3)$, $T \in \Gamma(G,k)$ and $e \in E(G)$. If following (i) or (ii) holds, then for $m=2,3$, G has edge-disjoint paths $P_1[x_1,x_2]$ and $P_2[x_1,x_m]$ such that $e \in E(P_1) \cup E(P_2)$ and $T \in \Gamma(G-\bigcup_{i=1}^{2} E(P_i),k-2)$.

(i) $e \in \partial_G(x_1,x_2)$,

(ii) $e \in \partial_G(x_1,y)$ for some $y \in V(G)-T$ with $d_G(y)=3$ and with $N_G(y)=\{x_1,x_2,x_3\}$.

Proof. Assume that (i) holds. By Theorem 1 if $m=2$, then G has a cycle C such that $e \in E(C)$ and $T \in \Gamma(G-E(C),k-2)$, and if $m=3$, then G has a path $P[x_2,x_3]$ such that $e \in E(P)$ and $T \in \Gamma(G-E(P),\ k-2)$.

Assume that (ii) holds. We may assume that G is 2-connected. If $d_G(x_3)=d>k$, then we replace x_3 by d vertices of degree k (Figure 4 gives an example with $d=8$ and $k=5$), producing a new graph G'. In G' we assign x_3 on $N_{G'}(y)-\{x_1,x_2\}$. If the result holds in G', then clearly the result holds in G, and so we may assume that $d_G(x_3)=k$. Let $f \in \partial_G(x_3)-\partial_G(y,x_3)$. By Lemma 3.1

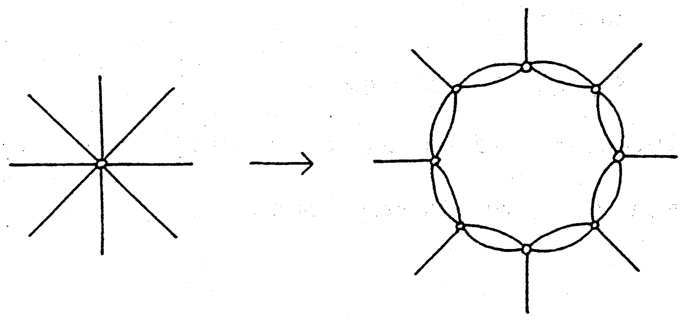

Figure 4.

G has a path $P[x_1,x_2]$ such that $x_3 \notin V(P), T \in \Gamma(G-E(P),k-2)$, $\{x_1,x_2,x_3\} \in \Gamma(G-E(P),k-1)$ and $\{x_i,x_3\} \in \Gamma(G-E(P)-f,k-1)$ $(i=1$ or $2)$. Then $y \notin V(P)$, because $d_G(x_3)=k$ and $d_G(y)=3$. Moreover $T \in \Gamma(G-E(P)-y,k-2)$. Thus the result follows.

Now we prove Theorem 2. We may assume that G is 2-connected, $d_G(x)=k$ for each $x \in T$ (see the proof of Lemma 3.5 and Figure 4, in this case we can assign x on any vertex of new $d_G(x)$ vertices of degree k) and that $d_G(y)=3$ for each $y \in V(G)-T$ (see Case 1 in the proof of Theorem 1). We proceed by induction on $|E(G)|$. If $|T| \lesssim 3$, then the results follows from Theorem 1. Thus let $|T| \gtrsim 4$.

Case 1. G has a nontrivial k-cut $\partial_G(X)=\{e_1,\ldots,e_k\}$ $(X \subseteq V(G))$ separating T.

We define H,K,u,v,T_H and T_K similarly as in the proof of Theorem 1. If $|X \cap T|=1$, then the results hold in K, and so in G. Thus let $|X \cap T| \gtrsim 2$ and $|T-X| \gtrsim 2$.

We require the following.

(3.3) If G has a nontrivial k-cut $\partial_G(Y)=\{f_1,\ldots,f_k\}$

$(Y \subseteq X)$ separating T, then we may assume that $(X-Y) \cap T \neq \emptyset$.

Proof. Assume $(X-Y) \cap T = \emptyset$. Let b_i (c_i) be the end of e_i (f_i) in Y $V(G)-X$ $(Y)(1 \leq i \leq k)$. We may assume that the graph obtained from $\langle X-Y \rangle_G$ by adding $b_1, \ldots, b_k, c_1, \ldots, c_k, e_1, \ldots, e_k, f_1, \ldots, f_k$ has edge-disjoint paths $P_1[b_1, c_1], \ldots, P_k[b_k, c_k]$. Let G' be the graph obtained from $G-(X-Y)$ by adding new edges $g_1, \ldots, g_k$, where g_i has ends b_i and $c_i (1 \leq i \leq k)$. Then $|E(G')| < |E(G)|$, and the results of Theorem 2 hold in G', and so in G. Now (3.3) is proved.

(3.4) If $|X-T|=2$ $(|T-X|=2)$, then we may assume that H (K) is $G(p,q)$ $(G(p',q'))$ for some integers p and q (p' and q').

Proof. Assume $|X \cap T|=2$. If H has a nontrivial k-cut $\partial_H(Y)$ $(Y \subseteq V(H)-u)$ separating T_H, then by (3.3) $(X-Y) \cap T \neq \emptyset$, and so $|T \cap Y|=1$. Then by taking Y instead of X the results of Theorem 2 hold. Thus we may assume that an end of each edge of H is in T_H. Hence the result easily follows.

We return to the proof of Theorem 2. By Lemma 3.5 we may assume the following.

(3.5) $\partial_G(a_1, a_i) = \emptyset$ $(i=2,m)$ and for each $y \in V(G)-T$, $\{a_1, a_2, a_m\} \not\subseteq N_G(y)$.

Let $a_1 \in X$.

(1) Now $|X-T|=|T-X|=2$. If $a_2 \in X$, then by (3.4) the result easily follows. Thus let $a_2 \in V(G)-X$. Since

$$p+q \geq (k+1)/2 \quad \text{and} \quad p'+q' \geq (k+1)/2,$$

for some $1 \leq i \leq k$, H has an elemental star S_1 containing a_1 and e_i and K has an elemental star S_2 containing a_2 and e_i. Then $T \in \Gamma(G-\bigcup_{i=1}^{2} E(S_i), k-1)$.

(2) Subcase 1-1. $\{a_2, a_m\} \subseteq X$.

H has required paths. If one of them passes through u, then we can deduce the result by using Lemma 3.1(3) on K.

Subcase 1-2. $\{a_2, a_m\} \subseteq V(G)-X$ and $|X \cap T|=2$.

Set $X \cap T=\{a_1, a_5\}$. By (3.4) H is G(p,q). Thus if following (3.6) or (3.7) holds, then the result follows.

(3.6) For some $e_i \in \partial_H(u, a_1)$, K has edge-disjoint paths $P_1[v, a_2]$ and $P_2[v, a_m]$ such that $e_i \in E(P_1) \cup E(P_2)$ and $T_K \in \Gamma(K-\bigcup_{i=1}^{2} E(P_i), k-2)$.

(3.7) For some $e_i, e_j \in \partial_H(u)-\partial_H(u, a_5)$, K has edge-disjoint paths $P_1[v, a_2]$ and $P_2[v, a_m]$ such that $\{e_i, e_j\} \subseteq E(P_1) \cup E(P_2)$ and $T_K \in \Gamma(K-\bigcup_{i=1}^{2} E(P_i), k-2)$.

If p=0, then $\partial_H(u, a_5)=\phi$, and so (3.7) follows. Thus let $p>0$. If $|T-X|=2$, then by (3.4) K is G(p′,q′), and so (3.6) follows. Thus let $|T-X|=3$ and m=3. Set $T-X=\{a_2, a_3, a_4\}$.

Subcase 1-2-1. K has nontrivial k-cut $\partial_K(Y)$ $(Y \subseteq V(K)-v)$ separating T_K.

By (3.3) We may let $|Y \cap T_K|=|T_K-Y|=2$. Let K_1 and K_2 be the graphs obtained from K by contructing Y and V(K)-Y to a vertex respectively. Then similarly as (3.4)

K_i is $G(p_i,q_i)$ for some integers p_i and q_i $(i=1,2)$

Let M be

$$\{\{x_1,x_2\} \subseteq V(K)-T_k \mid \partial_K(x_1,x_2) \neq \emptyset\},$$

and let M' be

$$\{\, x \mid \text{For some } N \in M,\ x \in N\}.$$

For each $N \in M$, $N \cap V(K_i) \neq \emptyset$ $(i=1,2)$, $d_{K-N}(a_j)=d_{K-N}(v)=k-1$ $(j=2,3,4)$ and $T_K \in \Gamma(K-N,k-1)$. If $k=|M|$, then $p_1=p_2=0$ and the result easily follows, and so let $k > |M|$. $K-M'$ is elemental for T_K and $k-|M|$.

Assume that $k-|M|$ is even and $K-M'$ is the graph obtained from four cycle by replacing each edge by $(k-|M|)/2$ parallel edges. For each cycle C of $K-M'$ such that $|V(C)|=|E(C)|=4$, we have $T_K \in \Gamma(G-E(C),k-2)$. If $\partial_G(a_1,a_4) \neq \emptyset$, then (3.6) follows, and if not, then by (3.5) a_1 is adjacent to p vertices of M'. If $|M| \geq 2$, then (3.6) follows. Thus assume $1 \geq |M| \geq p \geq 1$. Since $(k-|M|)/2 \geq (5-1)/2=2$, for some $1 \leq i < j \leq k$,

$$\{e_i,e_j\} \subseteq \partial_H(u)-\partial_H(u,a_5),$$

and K has a four cycle C such that $|V(C)|=|E(C)|=4$ and $\{e_i,e_j\} \subseteq E(C)$. Hence (3.7) follows.

By Lemma 3.4(2) we may assume that for each two vertices of T_K, $K-M'$ has an elemental star containing them. Set $a_0=v$, and for $i,j=0,2,3,4$, set

$$p_{i,j}=|\partial_K(a_i,a_j)|,$$

$$r_i=|\{\, x \in V(K)-T_K \mid N_K(x)=T_K-a_i\}|.$$

For $i,j=0,2,3,4$, since $|\partial_K(\{a_i,a_j\})| \geq k$,

$$p_{i,j} \leq (k-1)/2.$$

If a_1 is adjacent to a vertex of M' in G, then (3.6) follows. If for some $x \in V(G)-T$, $N_G(x)=\{a_1, a_i, a_4\}$ ($i=2$ or 3), then (3.6) follows. Thus and by (3.5) we may assume that

$$|\partial_G(a_1,a_4)|=p.$$

If $a_4 \in Y$, then (3.6) easily follows, and thus let $T_H-Y=\{a_0,a_4\}$. Since $p_{0,4} \geq |\partial_G(a_1,a_4)|=p > 0$, by Lemma 3.4(1) we have

$$p_{4,2} > 0,\ p_{4,3} > 0, \text{ or } r_0 > 0,$$

and

$$p_{0,2} > 0,\ p_{0,3} > 0, \text{ or } r_4 > 0.$$

If $r_0 > 0$, $r_4 > 0$, $p_{0,2} \cdot p_{3,4} > 0$, or $p_{0,3} \cdot p_{2,4} > 0$, then (3.6) follows (note that K_i is $G(p_i,q_i)$ for $i=1,2$) Thus we may assume that

(3.8) $p_{0,2} > 0$, $p_{2,4} > 0$ and $r_0=r_4=p_{0,3}=p_{3,4}=0$.

Assume $|M|=0$. Then

$$d_G(a_3)=p_{2,3}+r_2 \text{ and } p_{2,3} \leq (k-1)/2,$$

and so

(3.9) $r_2 \geq (k+1)/2 \geq p+1$.

By comparing $d_G(a_2)$ with $d_G(a_4)$ we have

$$p_{0,2}+p_{2,3}=p_{0,4}+r_2.$$

Thus

(3.10) $p_{0,2} > p_{0,4} \geq p$.

From (3.9) and (3.10), (3.7) follows.

Now we may let $|M| > 0$. Since $\{a_2,a_3\} \subseteq Y$, we have

$$|\partial_K(Y)|=k=d_K(a_2)+d_K(a_3)-2p_{2,3}-|M||$$
$$=2k-2p_{2,3}-|M|,$$

and so

$$2p_{2,3}+|M|=k.$$

Since $d_G(a_3)=k=p_{2,3}+r_2+|M|$,

$$r_2=p_{2,3}.$$

Since $d_G(a_3)=2r_2+|M|$, $d_G(a_4)=p_{0,4}+p_{2,4}+r_2+r_3+|M|$, and $p_{2,4}>0$ (by(3.8)), we have

(3.11) $r_2\geq a_{0,4}+1\geq p+1$.

By compairing $d_G(a_2)$ with $d_G(a_4)$, we have

$$p_{0,2}=p_{0,4}.$$

Thus

(3.12) $p_{0,2}+|M|\geq p+1$.

From (3.11) and (3.12), (3.7) follows.

Subcase 1-2-2. K has no nontrivial k-cut separating T_K.

We may assume that an end of each edge of K in T_K and K is elemental for T_K. The proof is similar as the case $|M|=0$ in the proof of Subcase 1-2-1.

Subcase 1-3. $\{a_2,a_m\}\subseteq V(G)-X$ and $|X\cap T|=3$.

Now m=3. By (3.4) K is $G(p',q')$. Set $X\cap T=\{a\ ,a_4,a_5\}$ If H has nontrivial k-cut $\partial_H(Y)$ $(Y\subseteq V(H)-u)$ separating T_H, then we may let $|Y\cap T_H|=2$. Then for Y or V(G)-Y instead of X Subcase 1-1 or Subcase 1-2 occurs. Thus we may assume that this is not the case and H is elemental for T_H. If following (3.13) or (3.14) holds, then the result follws.

(3.13) For some $e_i\in\partial_K(v)-\bigcup_{i=2}^{3}\partial_K(v,a_i)$, H has edge-disjoint paths $P_1[a_1,u]$ and $P_2[a_1,u]$ such that

$e_i \in E(P_1) \cup E(P_2)$ and $T_H \in \Gamma(H-\bigcup_{i=1}^{2} E(P_i), k-2)$.

(3.14) For $l=2$ or 3 and for some $e_i \in \partial_K(v,x_l)$ and $e_j \in \partial_K(v)-\partial_K(v,x_l)$, H has edge-disjoint paths $P_1[a_1,u]$ and $P_2[a_1,u]$ such that $\{e_i,e_j\} \subseteq E(P_1) \cup E(P_2)$ and $T_H \in \Gamma(H-\bigcup_{i=1}^{2} E(P_i), k-2)$.

Set $a_0=u$ and for $i,j=0,1,4,5$ set

$$p_{i,j}=|\partial_H(a_i,a_j)|,$$
$$R_i=\{x \in V(H)-T_H \mid N_H(x)=T_H-a_i\},$$
$$r_i=|R_i|.$$

By (3.5) $p_{0,1}=0$.

Assume $p_{1,4}=p_{1,5}=0$. If $r_0 \leq (k-1)/2$, then

$$r_4+r_5=d_G(a_1)-r_0 \geq (k+1)/2 \geq p'+1,$$

and so (3.13) or (3.14) follows. Thus let $r_0 \geq (k+1)/2$. Since $d_G(a_0)=p_{0,4}+p_{0,5}+r_1+r_4+r_5$ and $d_G(a_5)=p_{0,5}+p_{4,5}+r_0+r_1+r_4$, we have

$$p_{0,4}+r_5=p_{4,5}+r_0.$$

Hence

$$d_G(a_4)=k \geq p_{0,4}+r_0+r_5 \geq 2r_0 > k,$$

a contradiction.

Now we may let $p_{1,i}>0$ for $i=4$ or 5, say $i=4$. Since $p_{0,1}=0$ and by Lemma 3.4(3), we have

$$r_4+r_5 \geq 2.$$

For each $x \in R_4 \cup R_5$, if x is adjacent to a vertex of $V(K)-T_K$ in G, then (3.13) follows, thus assume that $\partial_G(x,a_i) \neq \emptyset$ ($i=2$ or 3). For each $x,y \in R_4 \cup R_5$, if $\partial_G(x,a_2) \neq \emptyset$ and $\partial_G(y,a_3) \neq \emptyset$, then (3.14) follows, thus assume that for $i=2$ or 3, $\partial_G(x,a_i)=\partial_G(y,a_i)=\emptyset$, say $i=3$, and that $r_4+r_5 \leq p'$.

Assume $r_4>0$. For some $e_i \in \partial_K(v)-\partial_K(v,a_2)$, e_i is incident to a_4 or a vertex of R_1 in G, because

$$p'+q' \geq (k+1)/2 > p_{0,5}.$$

Thus (3.14) follows.

Now we may assume that $r_4=0$, $r_5>0$ and $p_{1,5}=0$. Thus $p_{0,1}=p_{1,5}=r_4=0$, contrary to Lemma 3.4(1).

Subcase 1-4. $a_2 \in X$ and $a_m \in V(G)-X$.

now m=3.

Subcase 1-4-1. $|X \cap T|=2$.

By (3.4) $H=G(p,q)$, and by (3.5) $p=0$. Since $|T_K| \leq 4$, by induction K has a path $p[v,a_3]$ such that $T_K \in \Gamma(K-E(P),k-1)$. Let $e_1 \in E(P)$. H has an elemental star S_1 containing a_1 and e_1. Let S_2 be another elemental star of H. Then $T_H \in \Gamma(H-\bigcup_{i=1}^{2} E(S_i),k-2)$, and so the result follows.

Subcase 1-4-2. $|X \cap T|=3$ and $|T-X|=2$.

Assume that H has a nontrivial k-cut $\partial_H(Y)=\{f_1,\ldots,f_k\}$ $(Y \subseteq V(H)-u)$ separating T_H. Then we may assume that $|Y \cap T_H|=2$, $a_2 \in Y$ and $a_1 \in X-Y$. Let H_1 (H_2) be the graph obtained from H by contructing $V(H)-Y$ (Y) to a new new vertex u_1 (u_2). Then similarly as (3.4) H_i is $G(p_i,q_i)$ for some integers p_i and q_i $(i=1,2)$. If $p_2=0$, then the result easily follows. If $p_2>0$, then we may let $\{f_1,e_1\} \subseteq \partial_G(a_1)$ and we can easily deduce the result.

Now we may assume that H has no nontrivial k-cut separating T_H and H is elemental for T_H. Set $X \cap T=\{a_1,a_2,u,a_4\}$ and $T-X=\{a_3,a_5\}$. For a_1,a_2,u,a_4 instead of x_1,x_2,x_3,x_4, (a) or (b) of Lemma 3.4(3) holds. If (a) holds, then the result easily follows, thus assume that (b) holds. Since $|\partial_H(u)-\partial_H(u,a_2)|\geq(k+1)/2$ and $p'+q'\geq(k+1)/2$, for some $1\leq i\leq k$,

$$e_i\in \partial_H(u)-\partial_H(u,a_2) \text{ and } e_i\in \partial_K(v)-\partial_K(v,a_5),$$

and so the result follows.

Case 2. G has no nontrivial k-cut separating T.

We may assume that G is elemental for T. If $|T|=4$, then by Lemma 3.3 the result follows. Thus let $|T|=5$ and $m=3$. Set $T=\{a_1,a_2,a_3,a_4,a_5\}$ and for $1\leq i,j,l\leq 5$, set

$$p_{i,j}=|\partial_G(a_i,a_j)|,$$
$$R(i,j,l)=\{x\in V(G)-T \mid N_G(X)=\{a_i,a_j,a_l\}\},$$
$$r(i,j,l)=|R(i,j,l)|.$$

We require the following.

(3.15) For each distinct $1\leq i,j,l\leq 5$, G has an elemental star containig $\{a_i,a_j\}$ or $\{a_i,a_l\}$.

Proof. Assume that each elemental star of G does not contain $\{a_1,a_2\}$ nor $\{a_1,a_3\}$. Then

$$d_G(a_1)=p_{1,4}+p_{1,5}+r(1,4,5).$$

Since $p_{i,j}\leq(k-1)/2$ for each i,j, we have $r(1,4,5)>0$. Let F be a cut of G separating $\{a_1,a_4,a_5\}$ and $\{a_2,a_3\}$, then

$$|F|=d_G(a_4)+d_G(a_5)-(p_{1,4}+p_{1,5}+2r(1,4,5))< k,$$

a contradiction. Now (3.15) is proved.

We return to the proof of Theorem 2. By (3.5)

$$p_{1,2}=p_{1,3}=r(1,2,3)=0.$$

If $r(1,2,i)>0$ and $r(1,3,j)>0$ $(i,j=4$ or $5)$, then the result follows. Thus and by (3.15) we may assume that

$$r(1,2,4)>0 \text{ and } r(1,3,i)=0 \quad (i=4,5).$$

By (3.15)

$$p_{i,5}+r(i,5,2)+r(i,5,4)>0 \ (i=1,3).$$

If $p_{1,5}>0$, $p_{3,5}>0$, $r(1,5,2)\cdot r(3,5,4)>0$, or $r(1,5,4)\cdot r(3,5,2)>0$, then by Lemma 3.3 the result follows. Thus we may assume that for $(i,j)=(2,4)$ or $(4,2)$,

$$p_{1,5}=p_{3,5}=0,\ r(1,5,i)=r(3,5,i)=0,$$

and

$$r(1,5,j)\cdot r(3,5,j)>0.$$

Assume $r(1,5,2)=r(3,5,2)=0$. Then

$$d_G(x_1)=p_{1,4}+r(1,2,4)+r(1,4,5),$$

and

$$d_G(x_4)\geq p_{1,4}+r(1,2,4)+r(1,4,5)+r(3,4,5)>k,$$

a contradiction. Thus

$$r(1,5,4)=r(3,5,4)=0.$$

Since $r(1,2,5)>0$, by the same argument we have

$$p_{1,4}=p_{3,4}=0.$$

Thus

$$d_G(x_1)=r(1,2,4)+r(1,2,5)$$

and

$$d_G(x_2) \geq r(1,2,4)+r(1,2,5)+r(2,3,5) > k,$$

a contradiction.

4. PROOF OF THEOREM 3.

Suppose that $k \geq 1$ is an integer, G is a graph, $T=\{s_1,\ldots,s_k,t_1,\ldots,t_k\} \subseteq V(G)$ and $T \in \Gamma(G,k)$. We prove that if $|T|=3$, or if k is odd and $|T|=4$ or 5, then (1,1) holds by induction on k.

Assume $|T|=3$. By Theorem 1 G has a path $p[s_k,s_k]$ such that $T \in \Gamma(G-E(P),k-1)$. By induction for $k-1$, (1.1) holds in $G-E(P)$, and so for k, (1.1) holds.

Assume that $k \geq 5$ is odd and $|T|=4$ or 5. For some $1 \leq i < j \leq k$, if $|T|=4$, then

$$s_i=s_j \text{ or } t_j,$$

and if $|T|=5$, then

$$s_i=s_j \text{ or } t_j \text{ and } \{s_i,t_i\} \neq \{s_j,t_j\},$$

say for $i=k-1$ and $j=k$. By Theorem 2 G has edge-disjoint paths $P_1[s_{k-1},t_{k-1}]$ and $P_2[s_k,t_k]$ such that $T \in \Gamma(G-\bigcup_{i=1}^{2} E(P_i),k-2)$. By induction for $k-2$, (1.1) holds in $G-\bigcup_{i=1}^{2}E(P_i)$, and so for k, (1.1)holds in G.

Thus for integer $k \geq 1$,

$$\lambda'(k,3)=\lambda(k,3)=k,$$

and for odd integer $k \geq 1$,

$$\lambda'(k,4)=\lambda'(k,5)=k.$$

By Lemma 3.2 for odd integer $k \geq 1$,

$$\lambda(k,4)=\lambda(k,5)=k \text{ and } \lambda(k+1,4)=\lambda(k+1,5)=k+2.$$

Now Theorem 3 is proved.

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